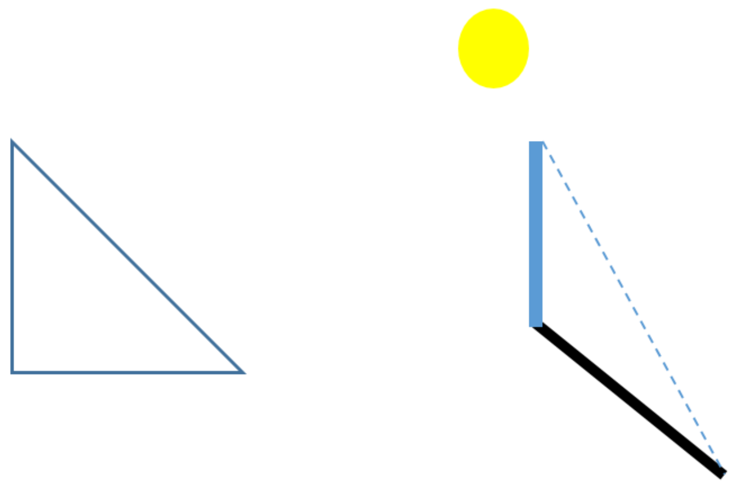
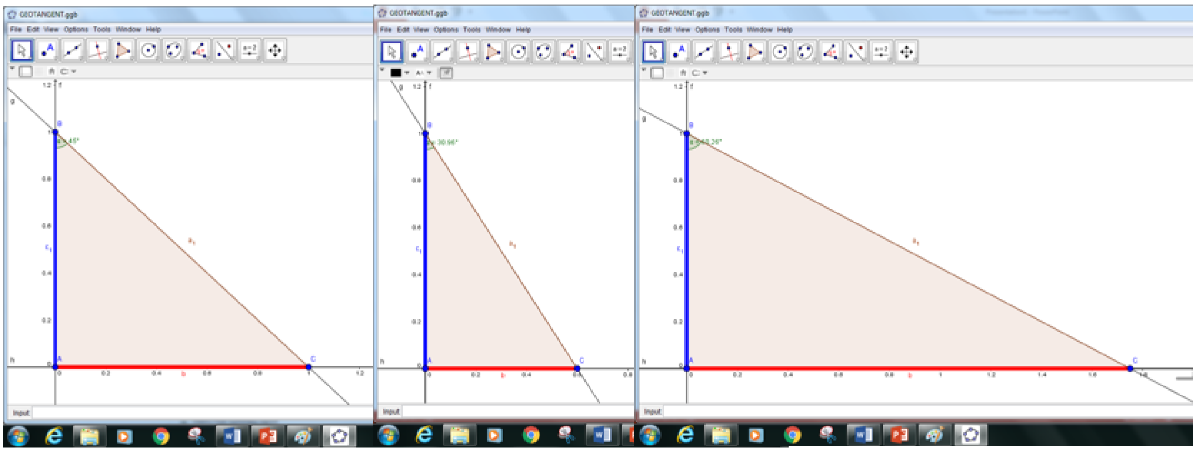
How important is perception and how does the way we are taught impact on it?

The catalyst for this blog came about as I was considering an alternative way of introducing basic trigonometry to learners and in particular I was reflecting on how looking at the mathematical structure of what was actually happening might (or might not) help understanding.

The model I came up with focused on what was the multiplying factor that changed the vertical line into the horizontal line. Not uniquely (see http://nrich.maths.org/6843) I realized I was considering the shadow cast by a vertical stick on to flat ground.



So in the case of a 45^o angle, the enlargement from the stick to the shadow was scale factor 1. Now this of course is the tangent value of 45 degrees, but what about other vales and so a dynamic model is needed.

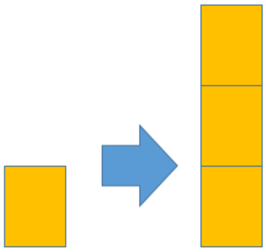


The Geogebra file used for the above images can be [downloaded from here.]()

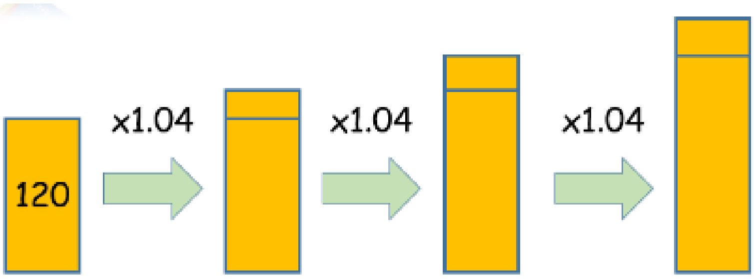
*Which illustrates the blue vertical line being **enlarged** to become the red horizontal line with varying scale factors above and below 1 but clearly dependent on the angle of the light casting the shadow.*

Now this is not new, but in reflecting on whether this would be an effective way of introducing the tangent ratio got me thinking about this whole question of multiplication as an enlargement, a topic that I've been mulling over for a while now.

This is a model I've used to represent multiplication previously…

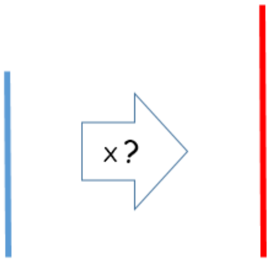


.. and although this represents a quantity multiplied by an integer (I often use 5 multiplied 3 times for this diagram) I also use a similar model for exemplifying non-integer multiplication, such as increase by a small percentage (as in this case of a compound interest model).

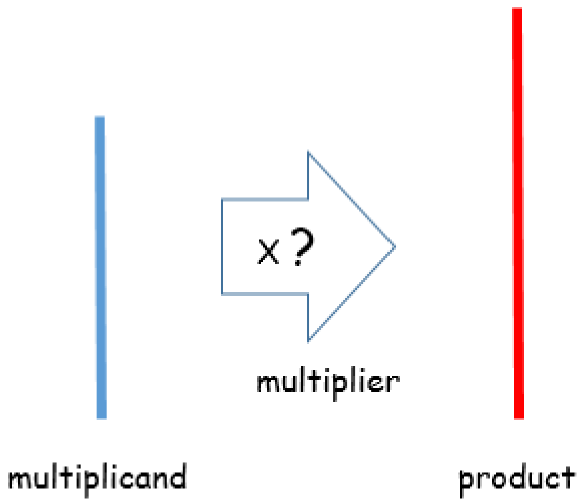


All very well but so what.

What I realized when thinking about my shadow diagram is that my own fluency wasn't embedded (mastered) enough to quickly arrive at the scale factor from the blue and red sticks, which were after all just the start and finish of a multiplication operation.



What I think I'm saying is, that my focus was too much on three numbers (the multiplicand, the multiplier and the product to use the correct terminology) for example 5x3 =15 or 120 x 1.05 = 126.



Instead of recognising that the multiplier must be equal to the

product ÷multiplicand

Which of course is true but I confess I had to stop and think about this rather than it be an obvious interpretation (something that I had personally observed in both Chinese and Japanese learners).

If $n \times \square = m$ then $\square = m / n$

And all the associated learning opportunities that goes with this..

If n = m then the multiplier must be 1

If n < m then the multiplier must be > 1

If n > m then the multiplier must be < 1

If n = 1 then the multiplier must be m

I knew all of this, so why didn't this come readily to me? And if I'm not the only one, is this a result of a lack of emphasis on basic multiplicative reasoning in my own education and more importantly if we do emphasise this (and other fundamental aspects of number) will the learners in our care benefit further down the line.

So my ramblings are now going to go down two paths.

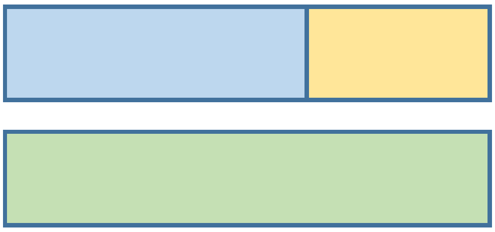
a) What models would improve this mastery of basic concepts?

b) How might teaching in later years be affected?

Focusing on the fundamental nature of multiplicative structure.

Bizarrely I'm going to illustrate my first thought with an additive structure, the part, part, whole.

My thoughts around this are, whether the language is important? Also if how we show the general structure both as a model or algebraically has an impact.



General algebraic structure for the part, part, whole model could simply be

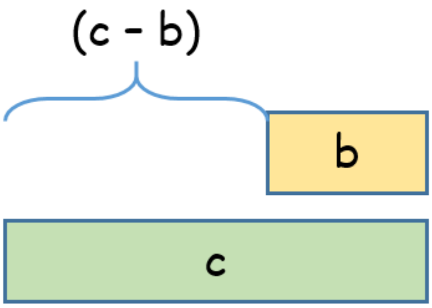
$$a + b = c$$

But in the case of addition should the emphasis be ,,

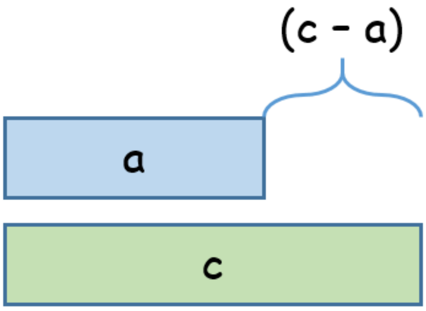
$$a + b = (a + b)$$

Now reading this it may sound crazy but I know the emphasis I've just put on the second "a plus b" and how I would say this to learners.

In the case of subtraction (take-away) the model may be



And the comparative model

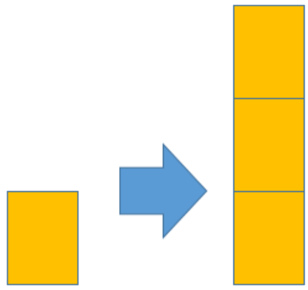


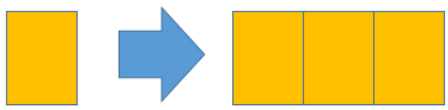
Let's consider multiplication.

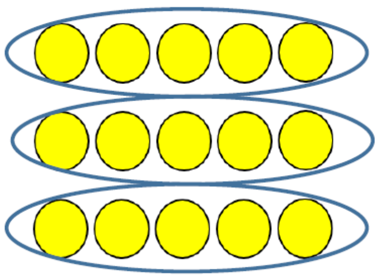
Our focus, I am implying is always on three numbers e.g. 3 x 5 = 15 rather than developing a mastery that enables learners to fluently see ..

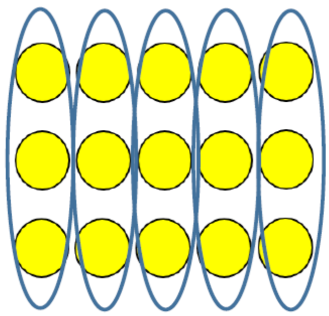
3 x 5 = □	*results in*	*□ = (3 x 5)*
4 x ○ = 28	*results in*	*○ = (28 ÷ 4)*
△ x 5 = 75	*results in*	*△ = (75 ÷ 5)*

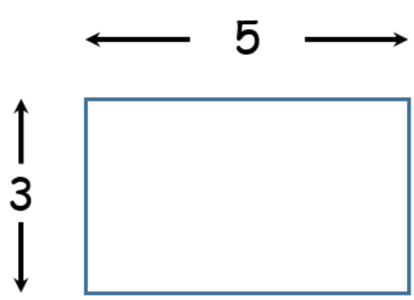
The link between my models and an array I would suggest are also significant but the linear enlargement model can be very useful (such as two-scale multiplicative reasoning) to discount it completely but are we effectively making the link between proportional enlargement, an array, rectangular area and a learner's calculation both mental and written methods?

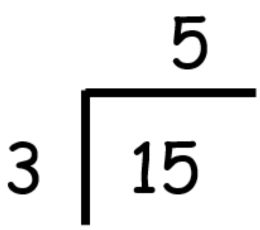












Should we have more emphasis on linking these models and the fluency of learners confidently understanding how to find any missing value from

- *Multiplicand, multiplier, Product*
- *Dividend, divisor, quotient.*
- *Length, width, area*

Or for the additive models,

- *Addend, addend, sum*
- *Subtrahend, minuend, difference*

Does our reluctance to use this language and generalise into well perceived rules (not tricks) shift a focus and possibly a perception that number sentences are something you learn as in number bonds or tables facts and that fluency is more than learning these same facts forwards and backwards but true fluency is seeing the connection between the, so any new set of facts can be derived from two numbers and the "third" doesn't even have to be expressed as a number but as a calculation (or possibly a fraction) this is a whole new ball game.

4 x □ = 3   results in □ = ¾

Times tables (shifting the emphasis?)

I guess its common practice to show learners how to use the tables square

	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	**24**	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Even for equivalent fractions

	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

But what if we emphasised the area shown by this table?

	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

You can download an active spreadsheet model of this

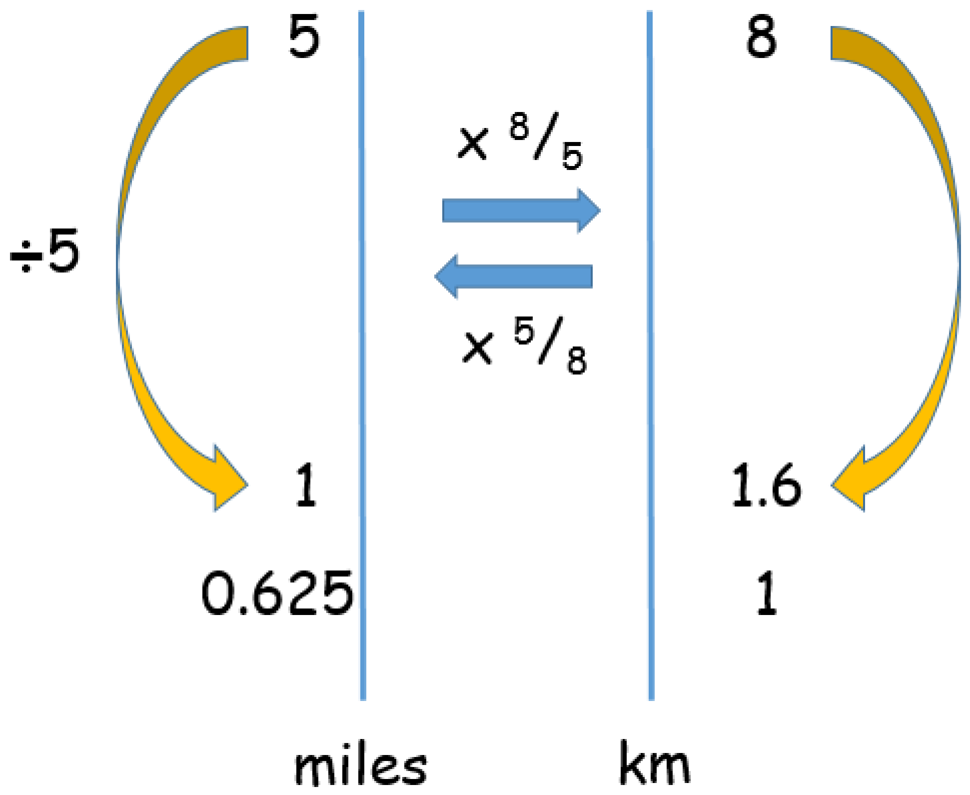
	1	2	3	4	5	6	7	8	9	10
1										
2										
3										
4										
5							35			
6										
7										
8										
9										
10										

$$5 \times 7 = 35$$

Also you can download a Geogebra file of something similar to play with

Impact on learners and teaching in later learning

In the multiplicative proportion model consider 8km = 5 miles



5 miles x ☐ *= 8 km*

So scale factor = 8/5 or 1.6

8km x ☐ *= 5 miles*

So scale factor = 5/8 or 0.625

Which reinforces the inverse of multiplication can be thought of as division or multiplication of the inverse.

Generally

N x ☐ *= M*

Therefore ☐ *=* $^{M}/_{N}$

Could also be applied to similar triangles (and other figures) as well as proportionality in general.

What about trigonometry

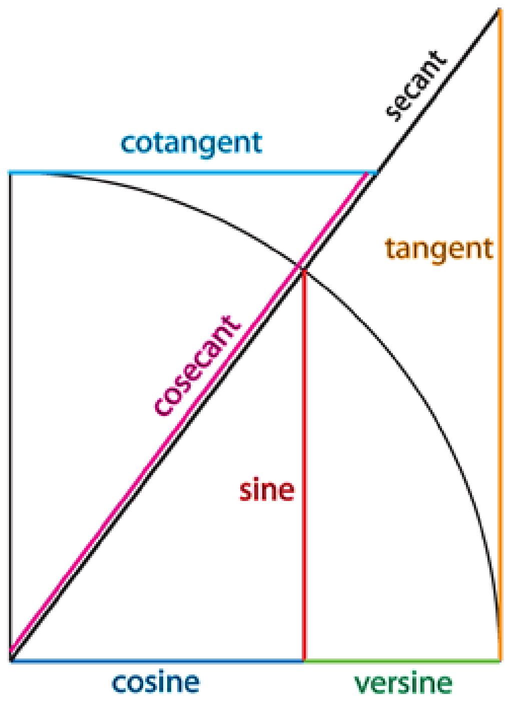
Referring back to my initial "shadow stick" diagram, If we then alter our perception of trigonometry from juts finding a missing angle or side to one of an enlargement of one side to another, say from the adjacent to opposite then we get this situation…

Adjacent x □ = Opposite

Where the scale factor in question must be Opposite ÷ Adjacent, which we know of course as the tangent ratio (thinking about it this way, the whole ratio idea makes much more sense too).

Of course there are a whole combination of "starting" sides to be enlarged into other sides so the need for a whole set of ratios each with a different name and if we use each of the starting sides as a value of 1 then the ratio in question will have the value of the final side.

You can [download a Geogebra file]() that may help students visualise this idea. The unit circle can now be introduced and if you look carefully at the diagram below (pinched for the above NRICH articles) you may like me learn a new named part and see why tangent is named as it is.



Psychosides

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